Int. J. Solids Structures Vol. 21, No. 2, pp. 133–143, 1985
Printed in the U.S.A.

0020-7683/85 $3.00 + .00

ON A THEORY FOR THERMO-VISCOELECTRIC MATERIALS

C. E. BEEVERS
Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, U.K.

and

R. E. CRAINE
Faculty of Mathematical Studies, Southampton University, Southampton SO9 5NH, Hampshire, U.K.

(*Received* 9 *June* 1983; *in revised form* 4 *January* 1984)

Abstract—This paper provides a sound thermodynamic foundation for the theory of thermo-viscoelectricity. The work is based on a non-equilibrium theory of thermodynamics described by Bree & Beevers [1] which generalises classical thermodynamics as proposed by Carathéodory [2]. A simple constitutive approach is outlined which can be used to discuss the well-known effects of Peltier, Seebeck and Thomson. A number of stability results for classical solutions to a class of initial-boundary value problems are deduced.

1. INTRODUCTION

Some years ago Bree & Beevers [1] constructed a new non-equilibrium theory of thermodynamics which extended Carathéodory's classical approach [2]. It is well known that the classical theory of thermodynamics cannot be rigorously applied to materials experiencing processes in which internal dissipation remains non-zero during quasi-static transitions (see, for example, Buchdahl [3]). Examples of such situations occur during the plastic deformation of an elastic-plastic material and in thermoelectricity. The new thermodynamic theory in [1] has been successfully applied to finite deformation plasticity [4, 5], however, and in this paper we apply the theory to thermoelectricity.

The governing mechanical and electromagnetic equations are presented in Section 2. The material is assumed to be non-polarisable, so no attempt is made to model piezoelectricity (see [6]), but finite viscoelastic deformations are allowed. It is the presence of these viscous effects which plays a vital role in our proof of the asymptotic stability of classical solutions.

A rigorous thermodynamic treatment of thermoelectricity is discussed in Section 3. This work provides a sound foundation for the practical effects described in a recent article by Shercliff [7]. A number of well-known simple cases are considered in the Appendix to this paper.

In the final two sections we consider classical solutions to the mixed initial-boundary value problem for isotropic thermo-viscoelectric materials within the framework of a linear theory. In Section 5, some stability and uniqueness theorems are proved. In particular, we establish conditions which ensure that the classical solution (for the linear problem) is asymptotically stable.

2. THE GOVERNING EQUATIONS

Let B be an isotropic continuous material which undergoes thermo-viscoelastic deformations. Heat conduction and electrical current flow can occur in the continuum B which occupies a finite region R of Euclidean three-space at time t. The region R has a smooth boundary ∂R. Relative to a fixed set of rectangular Cartesian axes a typical particle P of B has position $\mathbf{x}$ given by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \tag{2.1}$$

where $\mathbf{X}$ is the position of P relative to the fixed axes at time $t = 0$.

The velocity v and deformation F are defined by the equations

$$\mathbf{v} = \dot{\mathbf{x}}, \qquad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{2.2}$$

where a superposed dot denotes the material time derivative. The conservation of mass equation can then be written

$$J = \det \mathbf{F} = \rho_0/\rho, \tag{2.3}$$

ρ and ρ_0 being the densities in the current and initial configurations respectively. In addition conservation of linear and angular momentum yields the equations

$$\rho\dot{\mathbf{v}} = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f} + \mathbf{j} \times \mathbf{B} \tag{2.4}$$

and

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2.5}$$

In the above $\boldsymbol{\sigma}$ denotes the Cauchy stress, f represents the external body force per unit mass, j and B are the electrical current and magnetic field respectively, ∇ is the spatial gradient operator and the superscript ^{T} indicates the transpose.

Furthermore, if B is non-polarisable Maxwell's equations of electromagnetism reduce to

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla\cdot\mathbf{B} = 0, \tag{2.6}$$

$$-\epsilon\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\nu}\nabla \times \mathbf{B} = \mathbf{j}, \qquad \nabla\cdot\mathbf{E} = 0, \tag{2.7}$$

where E is the electric field and ϵ and ν are material constants.

3. A THERMODYNAMIC THEORY

The first law of thermodynamics asserts the existence of an internal energy function. We let u be the specific internal energy of B. For any part of B with volume V and smooth, regular boundary ∂V energy considerations require

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_V \rho(u + \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v})\,\mathrm{d}V\right\} = \int_V \{\rho(r + \mathbf{f}\cdot\mathbf{v}) + \mathbf{E}\cdot\mathbf{j}\}\,\mathrm{d}V + \int_{\partial V} (\mathbf{t}\cdot\mathbf{v} - \mathbf{q}\cdot\mathbf{n})\,\mathrm{d}A, \tag{3.1}$$

where r is the rate of supply of radiative heat to V per unit mass of V due to external sources, $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$ is the stress vector, q is the heat flux vector and n is the unit outward normal on ∂V.

On using eqn (2.4) and the divergence theorem the energy equation can be written

$$\int_V \{\rho\dot{u} - \rho r + \nabla\cdot\mathbf{q} - tr(\boldsymbol{\sigma}\cdot\nabla\mathbf{v}) - \boldsymbol{\epsilon}\cdot\mathbf{j}\}\,\mathrm{d}V = 0, \tag{3.2}$$

where $\boldsymbol{\epsilon}$ is the electromotive field given by

$$\boldsymbol{\epsilon} = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

Introduce the asymmetric Piola-Kirchhoff stress S and the work rate per unit mass $\dot{w}$ through the formulae

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \qquad \dot{w} = \frac{1}{\rho}\, tr(\boldsymbol{\sigma}\cdot\nabla\mathbf{v}), \tag{3.3}$$

respectively. It is then straightforward to establish the alternative expression

$$\dot{w} = \frac{1}{\rho_0}\, tr(\mathbf{S}\cdot\dot{\mathbf{F}}). \tag{3.4}$$

Thus, at a point within B, the energy balance reduces to

$$\dot{u} = r - \frac{1}{\rho}\nabla\cdot\mathbf{q} + \dot{w} + \frac{1}{\rho}\boldsymbol{\epsilon}\cdot\mathbf{j}. \tag{3.5}$$

A thermodynamic system Σ† is determined by any part P of a continuous material which consists always of the same material particles. When Σ is in equilibrium the stress in P depends upon a number of physical quantities $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ which we refer to as state variables. For a particular material the state variables are determined by experiment and they form an independent set of variables. For the material under investigation here we suppose that

$$\mathbf{S} = \mathbf{S}_E(\mathbf{F}, \theta) + \mathbf{S}_P, \tag{3.6}$$

where $\mathbf{S}_E$ is the equilibrium stress, $\mathbf{S}_P$ is that part of the stress due to viscous processes and θ is the empirical temperature. It follows from eqn (3.6) that $\boldsymbol{\xi} = (\mathbf{F}, \theta)$.

The non-equilibrium changes of Σ are simulated by associating with each Σ an equivalent standard system (e.s.s.). The e.s.s. has uniform properties which at any time coincide with the properties of Σ. So as Σ experiences non-equilibrium changes its e.s.s. undergoes a sequence of transitions from one equilibrium state to another (see [1] for a fuller discussion on this point). In this way the methods of classical thermodynamics can be preserved and may be rigorously applied to non-equilibrium systems.

From the first law of thermodynamics u is a variable of state and since $\boldsymbol{\xi}$ is an independent set of state variables then

$$u = u(\mathbf{F}, \theta). \tag{3.7}$$

On combining eqns (3.5)–(3.7) we obtain

$$\mathbf{P}(\boldsymbol{\xi})\cdot\dot{\boldsymbol{\xi}} = r + \dot{w}_p - \frac{1}{\rho}\nabla\cdot\mathbf{q} = h, \tag{3.8}$$

where

$$\mathbf{P} = \left(\frac{\partial u}{\partial \mathbf{F}} - \frac{1}{\rho_0}\mathbf{S}_E, \frac{\partial u}{\partial \theta}\right), \quad \rho\dot{w}_p = \boldsymbol{\epsilon}\cdot\mathbf{j} + \frac{1}{J}\, tr\{\mathbf{S}_p\cdot\dot{\mathbf{F}}\}. \tag{3.9}$$

In the general thermodynamic approach described in [1] it was pointed out that there are two types of equation within constitutive theory. There are equations of state involving state variables only and process equations which model the processes that arise in non-equilibrium systems. In order to specify these process equations it is usually necessary to include extra variables called process variables which may be gradients or rates of change of state variables or any other variables required to describe the processes. In this paper we are investigating three non-equilibrium processes: heat conduction, viscous dissipation and electrical current flow. The relevant process equa-

---

† Σ may be a single material particle or any number of material particles (not necessarily connected) as explained in [1].

tions are assumed to be

$$\mathbf{q} = \mathbf{q}(\mathbf{F}, \theta, \boldsymbol{\epsilon}, \mathbf{g}), \quad \mathbf{S}_p = \mathbf{S}_p(\mathbf{F}, \dot{\mathbf{F}}, \theta), \quad \mathbf{j} = \mathbf{j}(\mathbf{F}, \theta, \boldsymbol{\epsilon}, \mathbf{g}), \tag{3.10}$$

where $\mathbf{g} = \nabla\theta$ is the temperature gradient.

It is now appropriate to define some transitions of the system Σ. An adiabatic or a-transition is one which has

$$r = 0 \quad \text{in} \quad V, \qquad \mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial V. \tag{3.11}$$

An i_1-transition is one with

$$\dot{h} = 0 \quad \text{in} \quad V. \tag{3.12}$$

An i_2-transition requires $\theta = \theta(t)$ only and

$$\int_V \rho\dot{h}\,\mathrm{d}V = 0, \qquad \mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial V. \tag{3.13}$$

An (a, i)-transition of Σ is one any part of which is either an a-transition, an i_1-transition or an i_2-transition.

The statement of the second law of thermodynamics which generalises Carathéodory's classical statement is, quoting from [1]: In the neighbourhood of any given state of Σ there exist states which cannot be reached from the given state following an (a, i)-transition.

Using the above statement of the Second Law and Carathéodory's Mathematical Theorem it follows that there exist two variables of state T and s with

$$\dot{h} = T\dot{s}. \tag{3.14}$$

It can be shown that T is a positive function depending on the empirical temperature only and if $\mathrm{d}T/\mathrm{d}\theta > 0$ then T can be called the absolute temperature (see [1] for a fuller discussion). The function $s = s(\boldsymbol{\xi})$ is the specific entropy. The total entropy S of Σ is then defined by

$$S = \int_V \rho s\,\mathrm{d}V. \tag{3.15}$$

On combining eqns (3.8) and (3.14) it is easily demonstrated that

$$\mathbf{S}_E = \rho_0\frac{\partial\psi}{\partial\mathbf{F}}, \tag{3.16}$$

where $\psi = u - Ts$ is the specific free energy. Since by assumption $\dfrac{\mathrm{d}T}{\mathrm{d}\theta}$ is positive, θ can be replaced by T in the list of state variables $\boldsymbol{\xi}$ and within the process eqns (3.10). It is then straightforward to show that

$$s = -\frac{\partial\psi}{\partial T}. \tag{3.17}$$

Furthermore, as in [1], it follows that for all (a, i)-transitions of Σ we have $\dot{S} \geqslant 0$.

Now

$$\dot{S} = \int_V \rho\dot{s}\,\mathrm{d}V = \int_V \frac{\rho\dot{h}}{T}\,\mathrm{d}V = R_1 + \dot{S}_1, \tag{3.18}$$

where

$$R_1 = \int_V \frac{\rho r}{T}\,\mathrm{d}V - \int_{\partial V} \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}A, \quad \dot{S}_1 = \int_V \frac{1}{T}\left(\rho\dot{w}_p - \frac{\mathbf{q}\cdot\mathbf{g}}{T}\right)\mathrm{d}V, \tag{3.19}$$

and the inequality $\dot{S}_1 \geqslant 0$ holds when $R_1 = 0$. Thus, provided not all of the rates of change of the state variables appear in $\dot{S}_1$, the inequality

$$\int_V \rho\dot{s}\,\mathrm{d}V \geqslant \int_V \frac{\rho r}{T}\,\mathrm{d}V - \int_{\partial V} \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}A \tag{3.20}$$

is both a necessary and sufficient condition for consistency with thermodynamics. For the material under investigation $\dot{T}$ is absent from the process equations so from eqn (3.19) the necessary and sufficient pointwise restriction on the constitutive theory is

$$\frac{1}{J}\,tr(\mathbf{S}_p\cdot\dot{\mathbf{F}}) + \boldsymbol{\epsilon}\cdot\mathbf{j} - \frac{1}{T}\,\mathbf{q}\cdot\mathbf{g} \geqslant 0. \tag{3.21}$$

It should be emphasised that in our work inequalities (3.20) and (3.21) are *derived* results with primitive concepts like T and s well defined.

As pointed out by Buchdahl [3] the classical theory of thermodynamics is inapplicable to thermoelectricity since the term $\boldsymbol{\epsilon}\cdot\mathbf{j}$ does not disappear in the quasi-static limit. (Buchdahl refers to this situation as pseudo-static). We have shown above, however, that the introduction of the e.s.s. enables a thermodynamic theory to be formulated for these viscoelectric materials. Our theory is therefore more general than the classical approach and provides a firm foundation for thermo-viscoelectricity.

The simplifications which reduce our equations to those describing the well-known Peltier, Seebeck and Thomson effects are discussed in the Appendix.

4. THE INITIAL-BOUNDARY VALUE PROBLEM

In the final two sections of this paper we investigate the role of the viscous dissipative process on classical solutions of a class of initial-boundary value problems for the linearised version of the governing equations and introduce the displacement vector $\mathbf{u}$ through

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{4.1}$$

In order to derive the linear form of the governing equations for an isotropic, homogeneous viscoelectric material it is sufficient to put

$$\rho_0\psi = \tfrac{1}{2}\mu tr(\mathbf{e}\cdot\mathbf{e}) - b\tau tr\mathbf{e} - \frac{c}{2T_R}\tau^2, \tag{4.2}$$

where μ, b and c are material constants, T_R is the constant reference temperature, $\tau = T - T_R$ is the temperature deviation and the linear strain $\mathbf{e}$ is defined in terms of the displacement by

$$2\mathbf{e} = (\nabla\mathbf{u}) + (\nabla\mathbf{u})^T. \tag{4.3}$$

As pointed out in [1] the thermodynamic theory requires that $c > 0$ and as is customary we suppose that μ is strictly positive.

For the linear theory it follows from eqns (3.3), (3.6), (3.16), (3.17) and (4.2) that

$$\mathbf{S}_E = \sigma_E = \mu\mathbf{e} - b\tau\mathbf{I}, \qquad \rho_0 s = \frac{c\tau}{T_R} + btr\mathbf{e} \tag{4.4}^\dagger$$

where **I** is the unit tensor.

For our isotropic, homogeneous material there are also the three competing processes of viscous dissipation, heat conduction and electrical current flow. For the linear theory then the three process eqns (3.10) become

$$\mathbf{q} = -\kappa\mathbf{g} + \sigma\pi\mathbf{E}, \quad \mathbf{S}_p = \sigma_p = \lambda\dot{\mathbf{e}}, \quad \mathbf{j} = \sigma(\mathbf{E} - L\mathbf{g}), \tag{4.5}^\ddagger$$

where κ, σ, π, λ and L are all material constants. Some discussion of equations $(4.5)_{1,3}$ is given in the Appendix to this paper. The thermodynamic theory as embodied in inequality (3.21) provides the restrictions

$$\lambda \geqslant 0, \quad \sigma \geqslant 0, \quad \kappa \geqslant 0, \quad \frac{4\kappa}{\sigma T_R} \geqslant \left(L + \frac{\pi}{T_R}\right)^2. \tag{4.6}$$

In this work λ, σ and κ are assumed to be strictly positive.

The linear equations for this model can now be stated as follows:

$$\rho_0\ddot{\mathbf{u}} = \mu\nabla\cdot\mathbf{e} - b\nabla\tau + \lambda\nabla\cdot\dot{\mathbf{e}}, \tag{4.7}$$

$$c\dot{\tau} + bT_R tr\dot{\mathbf{e}} = \kappa\nabla\cdot\mathbf{g}, \tag{4.8}$$

$$\dot{\mathbf{B}} + \nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla\cdot\mathbf{B} = 0, \tag{4.9}$$

$$-\epsilon\dot{\mathbf{E}} + \frac{1}{\nu}\nabla \times \mathbf{B} = \mathbf{j}, \quad \nabla\cdot\mathbf{E} = 0, \tag{4.10}$$

where we have chosen $\mathbf{f} = \mathbf{0}$ and $r = 0$. A superposed dot denotes a time derivative. Equations (4.7) to (4.10) are valid for $\mathbf{x} \in R$ and $t > 0$. We impose the initial conditions

$$\left.\begin{aligned} &\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \tau(\mathbf{x}, 0) = \tau_0(\mathbf{x}),\\ &\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}), \end{aligned}\right\} \tag{4.11}$$

for $\mathbf{x} \in R$ and the boundary conditions

$$\left.\begin{aligned} &\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{x} \in \partial R_1; \quad \sigma\cdot\mathbf{n} = \mathbf{0}, \quad \mathbf{x} \in \partial R_2;\\ &\tau(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial R_3; \quad \mathbf{q}\cdot\mathbf{n} = 0, \quad \mathbf{x} \in \partial R_4;\\ &\qquad \mathbf{E} \times \mathbf{B}\cdot\mathbf{n} = 0, \quad \mathbf{x} \in \partial R; \end{aligned}\right\} \tag{4.12}$$

for $t > 0$. In eqn (4.12) **n** denotes the unit outward normal on ∂R and the surface regions ∂R_i satisfy $\partial R_1 \cap \partial R_2 = \phi = \partial R_3 \cap \partial R_4$ and $\partial R_1 \cup \partial R_2 = \partial R = \partial R_3 \cup \partial R_4$, where ϕ is the empty set. Let $\Gamma \equiv (\mathbf{u}, \tau, \mathbf{E}, \mathbf{B})$ then we say that Γ is a classical solution to our problem if it satisfies eqns (4.7)–(4.10) subject to conditions (4.11) and (4.12) and $\mathbf{u} \in C^2$, $\tau \in C^2$, $\mathbf{E} \in C$ and $\mathbf{B} \in C$ for all $(\mathbf{x}, t) \in R \times (0, \infty)$.

Next, form the inner product of eqn (4.7) with the velocity $\dot{\mathbf{u}}$ and integrate over the region R. After one application of the divergence theorem and conditions $(4.12)_{1,2}$ it

---

† Recalling definition $(3.3)_1$ we let $\sigma_E = J^{-1}\mathbf{F}\mathbf{S}_E$ and $\sigma_p = J^{-1}\mathbf{F}\mathbf{S}_p$ with $\sigma = \sigma_E + \sigma_p$.

‡ It should be noted that within linear theory the material and spatial time derivatives are identical.

can be shown that

$$\dot{K} + \dot{P} = \int_R \{b\tau tr\dot{\mathbf{e}} - \lambda tr(\dot{\mathbf{e}}\cdot\dot{\mathbf{e}})\}\, dV, \tag{4.13}$$

where

$$K = \tfrac{1}{2}\rho_0 \int_R \mathbf{v}\cdot\mathbf{v}\, dV, \qquad P = \tfrac{1}{2}\mu \int_R tr(\mathbf{e}\cdot\mathbf{e})\, dV. \tag{4.14}$$

Further, multiply eqn (4.8) by τ/T_R and integrate over R. Using the divergence theorem and conditions $(4.12)_{3,4}$ we have

$$\dot{H} = -\int_R \left\{b\tau tr\dot{\mathbf{e}} - \frac{1}{T_R}\mathbf{q}\cdot\mathbf{g}\right\} dV, \tag{4.15}$$

where

$$H = \frac{c}{2T_R}\int_R \tau^2\, dV. \tag{4.16}$$

Furthermore, we multiply eqn $(4.9)_1$ by $\frac{1}{\nu}\mathbf{B}$, eqn $(4.10)_1$ by $\mathbf{E}$ and subtract. Then, integrating over R and using condition $(4.12)_5$ we obtain

$$\dot{E} = -\int_R \mathbf{E}\cdot\mathbf{j}\, dV \tag{4.17}$$

where

$$E = E_1 + B_1, \quad E_1 = \tfrac{1}{2}\epsilon \int_R \mathbf{E}\cdot\mathbf{E}\, dV, \quad B_1 = \frac{1}{2\nu}\int_R \mathbf{B}\cdot\mathbf{B}\, dV. \tag{4.18}$$

On adding relations (4.13), (4.15) and (4.17) it follows that

$$\dot{F} = -\int_R \left\{\lambda tr(\dot{\mathbf{e}}\cdot\dot{\mathbf{e}}) + \mathbf{E}\cdot\mathbf{j} - \frac{1}{T_R}\mathbf{q}\cdot\mathbf{g}\right\} dV, \tag{4.19}$$

where $F = K + P + H + E$.

From the thermodynamic theory developed in the previous section the right hand side of eqn (4.19) is non-positive. Thus, the term on the left hand side of eqn (4.19) provides a natural Lyapunov function for this problem. Equation (4.19) generalises a much earlier result of Ericksen [8] for solutions of a class of initial-boundary value problems in thermoelasticity.

5. SOME STABILITY RESULTS

As noted at the end of the last section it is straightforward to prove:

Theorem 5.1

If a classical solution Γ exists then the positive definite measure $F(\Gamma)$ is bounded by its initial value.

Proof

The thermodynamic restriction of eqn (3.21) implies that eqn (4.19) can be written as

$$\dot{F} \leqslant 0. \tag{5.1}$$

Hence, $F(t) \leqslant F_0$ and the theorem is proved.

There is a simple corollary to this result:

Theorem 5.2 (*Uniqueness*)

If a classical solution Γ exists and $\mathbf{u}_0 = \mathbf{v}_0 = \mathbf{E}_0 = \mathbf{B}_0 = \mathbf{0}$ and $\tau_0 = 0$ then $\Gamma \equiv \mathbf{0}$.

Proof

With the given initial conditions $F_0 = 0$: hence

$$F(t) \leqslant 0. \tag{5.2}$$

Since F is a positive definite measure we deduce $\Gamma \equiv \mathbf{0}$.

Next, form the inner product of eqn (4.7) with the displacement $\mathbf{u}$ and integrate over R. Then, after application of the divergence theorem with conditions $(4.12)_{1,2}$ and definitions (4.14) we find that

$$\tfrac{1}{2}\ddot{U} - 2K + 2P + \frac{\lambda}{\mu}\dot{P} = \int_R b\tau tr e \,\mathrm{d}V, \tag{5.3}$$

where

$$U = \int_R \rho_0 \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}V. \tag{5.4}$$

Notice that for isothermal deformations ($\tau = 0$), for incompressible deformations ($tr e = 0$) or for zero thermal coupling ($b = 0$) the integral on the right hand side of eqn (5.3) vanishes identically. In each of these cases the following theorem follows more simply. Indeed from this point progress towards asymptotically stable results can take several routes which each depend on the specifics within a particular problem. To illustrate what can be proved we concentrate on just one of the possibilities. We will suppose that $\mathbf{B}$ is absent from the equations† and that ∂R_4 is empty.‡ It follows from Maxwell's eqns (4.9) and (4.10) that $\mathbf{E} = -\nabla\Omega$ where Ω is harmonic. We then prove:

Theorem 5.3

If ∂R_4 is empty and there exists a classical solution Γ with $\mathbf{B} \equiv \mathbf{0}$ then Γ is asymptotically stable.

Proof

On using eqns $(4.5)_{1,3}$ and $(4.18)_2$, eqn (4.19) can be written

$$\dot{F}_1 + 2\left(K_1 + H_1 + \frac{\sigma}{\epsilon}E_1\right) - \int_R \sigma\left(L + \frac{\pi}{T_R}\right)\mathbf{E}\cdot\mathbf{g}\,\mathrm{d}V = 0 \tag{5.5}$$

---

† Alternatively we might take $\mathbf{B}$ constant and assume that

$$\int_R \left\{\mathbf{E}\cdot\mathbf{j} - \frac{1}{T_R}\mathbf{q}\cdot\mathbf{g}\right\}\mathrm{d}V \geqslant \alpha_1 E + \alpha_2 H,$$

where α_1 and α_2 are positive constants.

‡ We could retain $\partial R_4 \neq \phi$ but assume that the strict inequality in eqn $(4.6)_4$ holds.

where

$$\begin{aligned} F_1 &= K + P + H + E_1, \\ K_1 &= \tfrac{1}{2}\lambda \int_R tr(\dot{e}\cdot\dot{e})\,\mathrm{d}V, \\ H_1 &= \frac{1}{2}\frac{\kappa}{T_R}\int_R \mathbf{g}\cdot\mathbf{g}\,\mathrm{d}V. \end{aligned} \tag{5.6}$$

Korn's inequality and Poincaré's inequality together imply the existence of a positive constant l_1 such that (see [9–12]):

$$K_1 \geqslant l_1 K, \qquad 2\frac{\lambda}{\mu}P \geqslant l_1 U. \tag{5.7}$$

A further Poincaré inequality reveals the existence of a constant $k_1 > 0$ with

$$H_1 \geqslant k_1 H. \tag{5.8}$$

With the assumptions that ∂R_4 is empty (i.e. $T = T_R$ on the whole boundary ∂R) and that $\mathbf{E} = -\nabla\Omega$ with Ω harmonic, the final term in eqn (5.5) can be shown to be identically zero. Then, combining the identities (5.3) and (5.5) with the inequalities $(5.7)_1$ and (5.8), we have

$$\dot{F}_1 + \frac{l_1}{4}\ddot{U} + \frac{l_1\lambda}{2\mu}\dot{P} + l_1(K + P) + 2\left(k_1 H + \frac{\sigma}{\epsilon}E_1\right) \leqslant \frac{l_1}{2}\int_R b\tau tre\,\mathrm{d}V. \tag{5.9}$$

On employing the weighted arithmetic-geometric mean inequality it is easily demonstrated that

$$\int_R b\tau tre\,\mathrm{d}V \leqslant \beta_1 P + \beta_2 H, \tag{5.10}$$

where β_1 can be made sufficiently small by an appropriate choice of the weight. Notice that in the derivation of eqn (5.9) we multiplied eqn (5.3) by the factor $\frac{1}{2}l_1$. This factor was chosen for convenience but any positive factor less than $\frac{1}{2}l_1$ could equally have been used. This is an important point in the subsequent analysis.

Combine eqns (5.9) and (5.10) and multiply by $\exp(\omega t)$ where ω is at our disposal. Then, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(F_1 + \frac{l_1}{4}\dot{U} + \frac{l_1\lambda}{2\mu}P\right)\mathrm{e}^{\omega t}\right\} + \left\{(l_1 - \omega)K + \left(l_1 - \frac{l_1\beta_1}{2} - \omega\right.\right. \\ &\qquad \left.\left. - \frac{\omega l_1\lambda}{2\mu}\right)P + (2k_1 - \tfrac{1}{2}l_1\beta_2 - \omega)H + \left(\frac{2\sigma}{\epsilon} - \omega\right)E_1 - \frac{\omega l_1}{4}\dot{U}\right\}\mathrm{e}^{\omega t} \leqslant 0. \end{aligned} \tag{5.11}$$

A further weighted arithmetic-geometric mean inequality has

$$\pm\dot{U} \leqslant \frac{2K}{w} + wU, \tag{5.12}$$

where w is the positive weight. With $w = l_1$ in inequality (5.12) we choose ω so that

$$\omega = \min\left\{\frac{2l_1}{3}, \frac{l_1\left(1 - \frac{1}{2}\beta_1\right)}{\left(1 + \frac{\lambda l_1}{\mu}\right)}, 2k_1 - \tfrac{1}{2}l_1\beta_2, \frac{2\sigma}{\epsilon}\right\}^{\dagger}, \tag{5.13}$$

then inequality (5.11) yields

$$F_1 + \frac{l_1}{4}\left(\dot{U} + \frac{2\lambda}{\mu}P\right) \leq J_0 e^{-\omega t} \tag{5.14}$$

where J_0 represents the initial value of $F_1 + \frac{l_1}{4}\left(\dot{U} + \frac{2\lambda}{\mu}P\right)$.

Another application of inequality (5.12) with $w = \tfrac{3}{4}l_1$ and the use of eqn $(5.7)_2$ provides

$$\tfrac{1}{3}K + P + H + E_1 + \frac{l_1^2}{16}U \leq J_0 e^{-\omega t}. \tag{5.15}$$

It is clear from the above that K, P, H, E_1 and U are all exponentially decaying as $t \to \infty$ and the theorem is proved.

In the absence of electromagnetic fields the above results would be valid for thermo-viscoelasticity (see Dafermos [13]). We also note that progress is possible for (non-zero) weak magnetic fields, anisotropic and inhomogeneous materials.

Acknowledgement—The authors are grateful to a referee for helpful comments on an earlier version of this paper.

REFERENCES

1. J. Bree and C. E. Beevers, Non-equilibrium thermodynamics of continuous media, *J. Non-Equil. Thermo.* **4,** 159–192 (1979).
2. C. Carathéodory, Untersuchungen über die Grundlagen der Thermodynamik. *Math. Ann.* **6,** 355 (1909).
3. H. A. Buchdahl, *The Concepts of Classical Thermodynamics,* Cambridge University Press (1966).
4. C. E. Beevers and J. Bree, A thermodynamic theory of isotropic elastic-plastic materials, *J. Mech. Stos.* **31,** 385–395 (1979).
5. J. Bree and C. E. Beevers, A thermodynamic treatment of elastic-plastic constitutive theory, *J. Non-Equil. Thermo.* **5,** 1–17 (1980).
6. C. E. Beevers and R. E. Craine, On the thermodynamics and stability of deformable dielectrics, *Appl. Sci. Res.* **37,** 241–256 (1981).
7. J. A. Shercliff, Thermoelectric magnetohydrodynamics, *J. Fluid Mech.* **91,** 231–251 (1979).
8. J. L. Ericksen, A thermo-kinetic view of elastic stability, *Int. J. Solids Struct.* **2,** 573–588 (1966).
9. T. W. Ting, Generalised Korn's inequality, *Tensor N.S.* **25,** 295–302 (1972).
10. J. E. Marsden and T. J. R. Hughes, *Topics in the Mathematical Foundations of Elasticity,* Nonlinear Analysis and Mechanics, Symposium Heriot-Watt University Vol. II, p. 30–251, Pitman (1978).
11. J. Nečas and I. Hlavaček, Mathematical theory of elastic and elasto-plastic bodies: An introduction. Elsevier, Amsterdam, Oxford, New York (1981).
12. C. B. Morrey, *Multiple Integrals in the Calculus of Variations,* Springer, Berlin (1966).
13. C. M. Dafermos, The mixed initial-boundary value problem for the equations of nonlinear one-dimensional viscoelasticity, *J. Diff. Eqns.* **6,** 71–86 (1969).

APPENDIX

To complete this paper we thought it would be appropriate to demonstrate how the familiar thermoelectric effects follow from a linearized version of our theory.

For simplicity consider a rigid continuum with $\mathbf{u} = \mathbf{0}$. In this case

$$\mathbf{v} = \mathbf{S} = \mathbf{0}, \tag{A1}$$

and we retain only the processes of heat conduction and electrical current flow. These processes may be

---

† The weighted arithmetic-geometric mean inequality (5.10) can always be used in a way which ensures that $\beta_1 < 2$. Then l_1 can be chosen so that $\beta_2 < 4k_1/l_1$. In this way the asymptotic result does not depend on the value of b, the coupling between the thermal and mechanical fields.

governed, as in eqn $(4.5)_{1,3}$, by

$$\mathbf{q} = -\kappa\mathbf{g} + \sigma\pi\mathbf{E}, \qquad \mathbf{j} = \sigma(\mathbf{E} - L\mathbf{g}), \tag{A2}$$

but here we allow κ, σ, π and L to depend on the absolute temperature T. The quantity σ is the electrical resistance. Combining eqn $(A2)_{1,2}$ leads to

$$\mathbf{q} = -k\mathbf{g} + \pi\mathbf{j}, \tag{A3}$$

where k is the thermal conductivity and π is the Seebeck coefficient. From inequality (3.21) the usual argument reveals that

$$k \geq 0, \qquad \sigma \geq 0, \qquad \frac{4k}{\sigma T} \geqslant \left(L - \frac{\pi}{T}\right)^2. \tag{A4}$$

For this simple model eqn $(4.4)_2$ reduces to $\rho_0 s = \dfrac{c\tau}{T_R}$ so that the energy eqn (3.8) together with equations $(A2)_2$ and (A3) provide

$$c\dot{T} = \rho_0 r + \nabla\cdot(k\mathbf{g}) + \Phi\mathbf{j}\cdot\mathbf{g} + \frac{1}{\sigma}\mathbf{j}\cdot\mathbf{j}, \tag{A5}$$

where the Thomson coefficient Φ is given by

$$\Phi = L - \frac{d\pi}{dT}. \tag{A6}$$

The four terms on the right-hand side of (A5) represent respectively heat due to radiation, conduction and the Thomson and Joule effects.

Now, let us consider some special cases of this constitutive theory. Suppose there is no temperature gradient, so $\mathbf{g} \equiv \mathbf{0}$. Then the quantity of heat Q and total current I flowing across a surface of area A with unit normal $\mathbf{n}$, defined by

$$Q = \int_A \mathbf{q}\cdot\mathbf{n}\, dA, \qquad I = \int_A \mathbf{j}\cdot\mathbf{n}\, dA, \tag{A7}$$

are related through

$$Q = \pi(T)I. \tag{A8}$$

This is the Seebeck effect.

The second special effect occurs when $\mathbf{j} = \mathbf{0}$. In this case equation $(A2)_2$ requires that

$$\mathbf{E} = L\mathbf{g}, \tag{A9}$$

and an electric field $\delta\epsilon = \mathbf{E}\cdot\delta\mathbf{r}$ may then be set up by a gradient of temperature only (the Peltier effect). It follows that

$$\frac{d\epsilon}{dT} = L; \tag{A10}$$

L being called the thermoelectric power. If, following Shercliff [7] it is assumed that

$$L = \frac{\pi}{T}, \tag{A11}$$

then the inequality $(A4)_3$ is automatically satisfied and substituting in (A10) we can deduce

$$\frac{d^2\epsilon}{dT^2} = \frac{1}{T}\frac{d\pi}{dT} - \frac{\pi}{T^2} = -\frac{\Phi}{T}. \tag{A12}$$

Equations (A10) and (A12) are Kelvin's first and second relations.